

## SYMMETRIC EXTENSIONS OF DIRICHLET OPERATORS

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Classical Dirichlet forms and operators are studied intensively in infinite-dimensional analysis and its applications (see [1] and references therein). Recently there have been studied supersymmetric Dirichlet forms and operators [2-3] because of numerous applications to quantum physics.

In this notice there is constructed and considered the extension of a classical Dirichlet operator in the space of symmetric differential forms.

The work consists of 3 parts. In the first part there're recalled some basic concepts and definitions concerning smooth measures on Hilbert spaces and corresponding Dirichlet forms and operators. In the part 2 there's constructed the extension of a classical Dirichlet operators and explicit form is given.

In the third part the extension is considered as a differential operator in the space of square integrated functions. There are given sufficient conditions for its essential self-adjointness in onedimensional case. Besides for the "supersymmetric part" of the operator conditions for self-adjointness in general situation are found.

1. Let  $\mathcal{H}$  be a real separable Hilbert space with a scalar product  $\langle \cdot, \cdot \rangle$  and a norm  $|\cdot|$  and  $\Phi' \supset \mathcal{H} \supset \Phi$  be a rigging of  $\mathcal{H}$  by nuclear Frechét space  $\Phi$  (densely and continuously embedded) and its dual space  $\Phi'$ . The duality between  $\Phi$  and  $\Phi'$ , which is given by the scalar product in  $\mathcal{H}$ , will be also denoted by  $\langle \cdot, \cdot \rangle$ .

Denote by  $\mathcal{FC}_b^\infty(\Phi)$  the set of all smooth cylinder functions on  $\Phi'$  with all derivatives bounded. For  $u \in \mathcal{FC}_b^\infty(\Phi)$  and  $\varphi \in \Phi$   $\nabla_\varphi u(\cdot) = \langle \nabla u(\cdot), \varphi \rangle$  is a directional derivative of a  $u(\cdot)$ .

Let  $\mu$  be a probability measure on  $(\Phi', \mathcal{B}(\Phi'))$  which is quasiinvariant under translations by all elements of  $\Phi$ . We also suppose that  $\mu$  has a logarithmic derivative  $\beta_\mu(\cdot)$  ( $\beta_\mu(\cdot) : \Phi' \rightarrow \Phi'$  measurable mapping (see [4])) and  $\beta_\mu(\cdot)$  is weakly differentiable for  $\mu$ -a.e.  $x \in \Phi'$ . Denote by  $R_\mu(x) = -\beta'_\mu(x)$ ,  $x \in \Phi'$  a family of self-adjoint (possibly unbounded) operators in  $\mathcal{H}$ , such that  $\Phi \subset D(R_\mu(\cdot)) \pmod{\mu}$ . We shall say that the measure  $\mu$  is uniformly log-concave (ULC) if  $\exists C > 0: \forall \varphi \in \Phi$   $\langle R_\mu(x)\varphi, \varphi \rangle \geq C|\varphi|^2$ .

In our further considerations we assume that the ULC measure  $\mu$  satisfies the following integrability conditions:

$$\forall \varphi \in \Phi \quad \int_{\Phi'} \beta_{\mu, \varphi}^2(x) d\mu(x) < \infty, \quad (1)$$

$$\forall \varphi \in \Phi \quad \int_{\Phi'} |R_\mu(x)\varphi|^2 d\mu(x) < \infty, \quad (2)$$

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where  $\beta_{\mu,\varphi}(x) := \langle \beta_\mu(x), \varphi \rangle$ ,  $x \in \Phi'$ .

On the domain  $D(H_\mu) = \mathcal{FC}_b^\infty(\Phi)$  define the differential operator  $H_\mu$  in  $L_2(\mu) \equiv L_2(\Phi', \mathcal{B}(\Phi'), \mu)$ :

$$(H_\mu u)(x) = -\Delta u(x) - \langle \beta_\mu(x), \nabla u(x) \rangle, \quad x \in \Phi',$$

where  $\Delta u(\cdot) = \text{Tr}_{\mathcal{H}} u''(\cdot)$ . Taking into account (1) we can see that the operator  $H_\mu$  is well defined in  $L_2(\mu)$ . This operator is called the Dirichlet operator corresponding to the measure  $\mu$ .

For  $u, v \in \mathcal{FC}_b^\infty(\Phi)$  define a positive symmetric form

$$\mathcal{E}_\mu(u, v) := (H_\mu u, v)_{L_2(\mu)} = \int_{\Phi'} \langle \nabla u(x), \nabla \bar{v}(x) \rangle d\mu(x).$$

The form  $\mathcal{E}_\mu$  is obviously closable and its closure is a classical Dirichlet form.

2. Consider the space

$$\Gamma_\mu(\mathcal{H}) := L_2(\mu) \otimes \Gamma(\mathcal{H}) = \bigoplus_{n=0}^{\infty} L_2(\mu) \otimes \Gamma^n(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \Gamma_\mu^n(\mathcal{H}),$$

where  $\Gamma(\mathcal{H})$  is a symmetric (bosonic) Fock space and  $\Gamma^n(\mathcal{H})$ ,  $n \geq 0$  are  $n$ -particle subspaces ( $\Gamma^0(\mathcal{H}) = \mathbb{C}$ ). The following representation of  $\Gamma_\mu(\mathcal{H})$  is helpful:  $\Gamma_\mu(\mathcal{H}) = L_2(\Phi' \rightarrow \Gamma(\mathcal{H}), \mu)$ , for  $F \in \Gamma_\mu(\mathcal{H})$   $\|F\|^2 = \int_{\Phi'} \|F(x)\|_{\Gamma(\mathcal{H})}^2 d\mu(x)$ .

For  $n \geq 0$  the set  $D_n(\Phi) := l.s.\{u_n(x) = u(x)\varphi_1 \hat{\otimes} \dots \hat{\otimes} \varphi_n : u \in \mathcal{FC}_b^\infty(\Phi), \varphi_k \in \Phi\}$  is dense in  $\Gamma_\mu^n(\mathcal{H})$ . Then the set  $D(\Phi) = \{(u_0, u_1, \dots, u_N, 0, \dots, 0) : u_k \in D_k(\Phi)\}$  is dense in  $\Gamma_\mu(\mathcal{H})$ .

For any  $n \geq 0$  introduce the linear operator  $\delta_n : \Gamma_\mu^n(\mathcal{H}) \rightarrow \Gamma_\mu^{n+1}(\mathcal{H})$  with the domain  $D_n(\Phi)$ :

$$(\delta_n u_n)(x) = \sqrt{n+1} \nabla u(x) \hat{\otimes} \varphi_1 \hat{\otimes} \dots \hat{\otimes} \varphi_n$$

Using the operator  $\delta_n$ ,  $n \geq 0$  on the domain  $D(\Phi)$  define a linear operator in  $\Gamma_\mu(\mathcal{H})$ :

$$(\delta u)_n = \delta_{n-1} u_{n-1}, \quad (\delta u)_0 = 0 \tag{3}$$

A direct computation yields:

$$(\delta_n^* u_{n+1})(x) = -\frac{1}{\sqrt{n+1}} \sum_{j=1}^{n+1} (\nabla_{\varphi_j} u(x) + \beta_{\mu, \varphi_j}(x) u(x)) \varphi_1 \hat{\otimes} \dots \hat{j} \dots \hat{\otimes} \varphi_{n+1}.$$

Analogously (3) define the operator  $\delta^*$  in  $\Gamma_\mu(\mathcal{H})$  with the domain  $D(\Phi)$ :

$$(\delta^* u)_n = \delta_n^* u_{n+1}.$$

Finally, on the domain  $D(\Phi)$  define a linear operator in  $\Gamma_\mu(\mathcal{H})$

$$\Delta_\mu = \delta^* \delta + \delta \delta^*.$$

Obviously  $\Delta_\mu$  is a positive symmetric operator.

**Theorem 1.** *On  $D(\Phi)$*

$$\Delta_\mu = H_\mu \otimes \mathbf{1} + \mathbf{1} \otimes d\Gamma(R_\mu(\cdot)) + \mathbf{A}_\mu, \quad (4)$$

where  $d\Gamma(R_\mu(\cdot))$  is the second quantization of  $R_\mu(\cdot)$  and  $\mathbf{A}_\mu$  is a positive symmetric operator. Its explicit form will be given below. As the conditions (1) and (2) are fulfilled all the operators in (4) are well defined in  $\Gamma_\mu(\mathcal{H})$ .

The operator  $\Delta_\mu$  is symmetric extension of Dirichlet operator  $H_\mu$ .

Point out that in [3] the extension was the same as in (4) just without the operator  $\mathbf{A}_\mu$ .

3. Let us make a transition from  $\Gamma_\mu(\mathcal{H})$  to  $L_2(\mu) \otimes L_2(\gamma)$  using the Segal isomorphism,  $\gamma$  is the standard Gaussian measure on  $\Phi'$ . We shall obtain (see [5], ch. 6])

$$\begin{aligned} S\Gamma_\mu(\mathcal{H}) &= L_2(\mu) \otimes L_2(\gamma), \\ SD(\Phi) &= D = l.s.\mathcal{FC}_b^\infty(\Phi) \otimes \mathcal{P}(\Phi'), \\ \Delta_\mu &= S\Delta_\mu S^{-1} = H_\mu \otimes \mathbf{1} + \mathbf{1} \otimes H_{\gamma, R_\mu(\cdot)} + \mathbf{A}_\mu = H_{\mu, \gamma} + \mathbf{A}_\mu, \end{aligned}$$

where  $S = \mathbf{1} \otimes I$ ,  $\mathcal{P}(\Phi')$  is the set of all continuous polynomials on  $\Phi'$ ,  $H_{\gamma, R_\mu(\cdot)}$  is the Dirichlet operator of the measure  $\gamma$  with coefficient operator  $R_\mu(\cdot)$ , see [5, ch. 6], and the operator  $\mathbf{A}_\mu$  is defined by its action on functions  $u(\cdot)p(\cdot) \in D$ :

$$\begin{aligned} (\mathbf{A}_\mu u p)(x, y) &= Tr_H u''(x)p''(y) + \langle p''(y)\beta_\mu(x), \nabla u(x) \rangle \\ &\quad - \langle u''(x)y, \nabla p(y) \rangle - \langle \beta_\mu(x), y \rangle \langle \nabla u(x), \nabla p(y) \rangle, \quad x, y \in \Phi', \end{aligned}$$

as  $\beta_\gamma(y) = -y$ ,  $y \in \Phi'$ .

We can see that  $\mathbf{A}_\mu$  is the fourth order differential operator of a complex structure. That's why one managed to find the condition for essential self-adjointness of the operator  $\Delta_\mu$  just in case, if  $\Phi' = \mathcal{H} = \Phi = \mathbb{R}$ . We have  $\Delta_\mu = H_\mu \otimes \mathbf{1} + \mathbf{1} \otimes H_{\gamma, R_\mu(\cdot)} + 2H_\mu \otimes H_\gamma$  on  $C_0^\infty(\mathbb{R}) \otimes \mathcal{P}(\mathbb{R})$ . The following result is true (see [6]).

**Theorem 2.** *Let  $f(\cdot)$  be the density of the measure  $\mu$ . If  $f \in C^1(\mathbb{R})$  and  $\forall x \in \mathbb{R}$   $f(x) > 0$ , the operator  $\Delta_\mu$  is essentially self-adjoint on  $C_0^\infty(\mathbb{R}) \otimes \mathcal{P}(\mathbb{R})$ .*

In infinite-dimensional case because of the complex structure of the operator  $\mathbf{A}_\mu$  there're no satisfactory conditions for essential self-adjointness of  $\Delta_\mu$ . But it's possible to find such conditions for the operator  $H_{\mu, \gamma}$ .

It appears that  $H_{\mu, \gamma}$  may be considered as the Dirichlet operator of a certain perturbed measure on  $(\Phi' \times \Phi', \mathcal{B}(\Phi' \times \Phi'))$ . That's why in order to prove essential self-adjointness of  $H_{\mu, \gamma}$  one can use one of numerous theorem, concerning classical Dirichlet operators (see e.g. [7]).

Let  $\Phi' \supset \mathcal{H}_- \supset \mathcal{H}$ ,  $\mu(\mathcal{H}_-) = \gamma(\mathcal{H}_-) = 1$ . Denote by  $g(x, y) := \|R_\mu^{-1/2}(x)y\|_-$  and  $h(x, y) := \|(R'_\mu(x)R_\mu^{-1/2}(x)y)R_\mu^{-1/2}(x)y\|_-$ ,  $x, y \in \mathcal{H}_-$ , where  $R'_\mu(\cdot)$  is the weak derivative of  $R_\mu(\cdot)$ . Now we are able to formulate

**Theorem 3.** *Assume that  $\beta_\mu(\cdot) \in C_{b,loc}^3(\mathcal{H}_-, \mathcal{H}_-)$ ,  $\|\beta_\mu(\cdot)\|_- \in L_2(\mu)$ ,  $\{g, h\} \subset L_2(\mu \times \gamma)$ . Then the operator  $H_{\mu, \gamma}$  is essentially self-adjoint on  $D$ .*

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